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# Regularity of solutions of initial boundary value problems for symmetric hyperbolic systems with boundary characteristic of constant multiplicity

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## 1. Introduction.

Let  $\Omega$  be a bounded open set in  $\mathbf{R}^n$ ,  $n \geq 2$ , with smooth boundary  $\Gamma$ . We consider the initial boundary value problem for the system of linear partial differential equations of first order

$$(1.1) \quad \begin{cases} \sum_{j=0}^n A_j \partial_j u + A_{n+1} u = F & \text{in } [0, T] \times \Omega \\ Qu = 0 & \text{on } [0, T] \times \Gamma \\ u(0) = f & \text{on } \Omega, \end{cases}$$

where  $x_0$  is the time variable, sometimes written as  $t$ ,  $\partial_j = \partial/\partial x_j$ ,  $0 \leq j \leq n$ , and the coefficients  $A_j$ ,  $0 \leq j \leq n+1$ , and  $Q$  are  $l_0 \times l_0$  complex matrix-valued functions on  $[0, T] \times \bar{\Omega}$  and  $\Gamma$  respectively.

We assume that (1.1) is a symmetric system with a maximal nonnegative boundary condition in the sense of Friedrichs [5] and Lax-Phillips [8]. The matrix  $\sum_{j=1}^n \nu_j A_j$  defined on  $[0, T] \times \Gamma$ , where  $\nu = {}^t(\nu_1, \dots, \nu_n)$  is the unit outward normal to  $\Gamma$ , is called the boundary matrix. When the boundary matrix is regular everywhere on  $[0, T] \times \Gamma$ , the problem (1.1) is called non-characteristic and in the other cases characteristic. There are many studies on the strong solution in the sense of Friedrichs in both the non-characteristic and characteristic cases ([5], [8] and [8], [16], [17], etc. respectively). In this paper we are interested in the higher order regularity of the strong solution to the characteristic problem.

The strong solution to the non-characteristic problem evolves continuously in the usual Sobolev space just like the solution to the Cauchy problem ([18], [27]). Some characteristic equations enjoy the same property thanks to their special structure ([7], [10], [11]). This is not always true of all the characteristic problems, as illustrated by several equations including the one of ideal magneto-hydrodynamics ([10], [13], [26]). Hence, we are forced to introduce some other function spaces than the usual Sobolev spaces in handling the higher order regularity of solutions to the characteristic problem of a general form.

A few spaces have been proposed when the boundary matrix is of constant rank. Rauch [16] proved that the strong solution and its derivatives in  $t$  evolve continuously in the function spaces in which only the regularity of tangential derivatives in the  $L^2$ -sense is taken into account. This result, referred to as the tangential regularity, is not available for solving quasilinear problems because the function space lacks several properties indispensable to nonlinear analysis. Yanagisawa–Matsumura [29] introduced some weighted Sobolev spaces in which the regularity of normal derivatives is appropriately considered and succeeded in solving the equation of ideal magneto–hydrodynamics. Ohno–Shizuta–Yanagisawa [15] handled the equation of a general form using the same function spaces. We note that the weighted Sobolev space, denoted by  $H_*^m(\Omega)$ , was first introduced by Chen Shuxing [4] in the study of a class of quasilinear hyperbolic systems.

The continuation of solutions in the weighted spaces needs further improvements on the known results. Shizuta–Yabuta [22] presented a compatibility condition for the solution to lie in  $H_*^m(\Omega)$  but failed to find the solution in this class. A proof of this part was given by Secchi [20], [21]. His idea is raising the regularity of the strong solution one by one up to the desired order. To obtain the tangential regularity, for instance, he considered the equations for the tangential derivatives of the solution. With some equations added they form a system of first order. Secchi expected the derivatives as smooth as the solution of the system and tried to solve it. The claim is that the solution is the fixed point of a contraction map sending an element of a certain metric space to the solution of the equation in which the unknown function of the system is partially replaced by the element. His plan, however, seems not to work well here, for some other hypotheses on the structure of the coefficient matrices are required than the assumptions to solve this equation for all the elements of the metric space.

In fact, the conclusion itself is true and the proof is straightforward as we will show in this paper. Unlike [20], [21] we pick up the system of equations for the tangential derivatives. By taking the degeneracy of the boundary matrix into account carefully the system is just of the same form as (1.1). Hence, we have only to concentrate on the study of the first order regularity of strong solutions. The energy method suffices for our argument. It is also used to obtain the regularity of the normal derivatives of the solution. No space with negative norm is involved as compared with [20], [21].

We plan this paper as follows. In section 2 the definitions of several function spaces and their basic properties are given. In section 3 we present the assumptions and the statement of the main results. Section 4 is devoted to the proof of the existence of solutions of first order regularity. The next two sections treat the higher order regularity of solutions. All the technicalities are collected in Appendix.

## 2. Notation and function spaces.

$\mathbf{R}$  and  $\mathbf{C}$  denote the fields of real and complex numbers respectively.  $\mathbf{N}$  is the set of natural numbers and  $\mathbf{Z}_+$  the set of nonnegative integers.

Let  $E$  be a Banach space,  $m \in \mathbf{Z}_+$ , and  $1 \leq q \leq \infty$ . We set several function spaces with values in  $E$  as follows. For a compact interval  $I$  we denote the space of  $m$  times continuously differentiable functions on  $I$  by  $C^m(I; E)$ .  $C_w^m(I; E)$  is the space of  $m$  times weakly continuously differentiable functions on  $I$ . Let  $I$  be an open interval.  $L^q(I; E)$  is the  $L^q$ -space with respect to the Lebesgue measure on  $I$ .  $W_q^m(I; E)$  is the Sobolev space in  $I$  of order  $m$ :

$$\{u \in L^q(I; E); \text{distributional derivatives } \partial^j u \in L^q(I; E), 0 \leq j \leq m\}.$$

These spaces are equipped with the natural norms and are Banach spaces.

Let  $\Omega$  be a bounded open set in  $\mathbf{R}^n$ ,  $n \geq 2$ , with smooth boundary  $\Gamma$ .  $H^m(\Omega)$ ,  $m \in \mathbf{Z}_+$ , is the usual Sobolev space in  $\Omega$  of order  $m$ . We see  $H^0(\Omega) = L^2(\Omega)$ . We introduce the subspaces  $H_*^m(\Omega)$  and  $H_{**}^m(\Omega)$  of  $L^2(\Omega)$  which play crucial roles in this paper. Also the space  $H_{\text{tan}}^m(\Omega)$  is given. We begin with the notion of tangential vector fields. Let  $\Lambda$  be a  $C^\infty$ -vector field on  $\bar{\Omega}$ .  $\Lambda$  is said tangential if for any  $C^\infty$ -function  $u$  on  $\bar{\Omega}$  vanishing on  $\Gamma$  we have  $\Lambda u = 0$  on  $\Gamma$ .

**Definition 1.** Let  $m \in \mathbf{N}$ .  $H_*^m(\Omega)$  is the set of a function in  $L^2(\Omega)$  such that all the distributions which result from operating  $j$  tangential vector fields and  $k$  vector fields to the function lie in  $L^2(\Omega)$  provided

$$(2.1) \quad 0 \leq j + 2k \leq m.$$

The spaces  $H_{**}^m(\Omega)$  and  $H_{\text{tan}}^m(\Omega)$  are defined by putting the conditions

$$(2.2) \quad 0 \leq j + 2k \leq m + 1, \quad 0 \leq j + k \leq m,$$

$$(2.3) \quad 0 \leq j \leq m, \quad k = 0,$$

in place of (2.1) respectively. We define  $H_*^0(\Omega) = H_{**}^0(\Omega) = H_{\text{tan}}^0(\Omega) = L^2(\Omega)$ .

In a region apart from the boundary  $\Gamma$  elements of these spaces behave like functions in  $H^m(\Omega)$ . For describing the behavior of the elements near  $\Gamma$  it is convenient to introduce some standard function spaces. Let  $\mathbf{R}_+^n = \{x; x_n > 0\}$ . For  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$  we put

$$\partial_{\text{tan}}^\alpha = \partial_1^{\alpha_1} \cdots \partial_{n-1}^{\alpha_{n-1}} (x_n \partial_n)^{\alpha_n}.$$

**Definition 2.** Let  $m \in \mathbf{N}$ .  $H_*^m(\mathbf{R}_+^n)$  is the set of  $u \in L^2(\mathbf{R}_+^n)$  satisfying  $\partial_{\text{tan}}^\alpha \partial_n^k u \in L^2(\mathbf{R}_+^n)$ ,  $|\alpha| + 2k \leq m$ .  $H_{**}^m(\mathbf{R}_+^n)$  is the set of  $u \in L^2(\mathbf{R}_+^n)$  satisfying  $\partial_{\text{tan}}^\alpha \partial_n^k u \in$

$L^2(\mathbf{R}_+^n)$ ,  $|\alpha|+2k \leq m+1$ ,  $|\alpha|+k \leq m$ .  $H_{\tan}^m(\mathbf{R}_+^n)$  is the set of  $u \in L^2(\mathbf{R}_+^n)$  satisfying  $\partial_{\tan}^\alpha u \in L^2(\mathbf{R}_+^n)$ ,  $|\alpha| \leq m$ . We define  $H_*^0(\mathbf{R}_+^n) = H_{**}^0(\mathbf{R}_+^n) = H_{\tan}^0(\mathbf{R}_+^n) = L^2(\mathbf{R}_+^n)$ .

$H_*^m(\mathbf{R}_+^n)$ ,  $H_{**}^m(\mathbf{R}_+^n)$  and  $H_{\tan}^m(\mathbf{R}_+^n)$  are Hilbert spaces with respective norms

$$\begin{aligned} |u|_{H_*^m(\mathbf{R}_+^n)} &= \left\{ \sum_{|\alpha|+2k \leq m} |\partial_{\tan}^\alpha \partial_n^k u|_{L^2(\mathbf{R}_+^n)}^2 \right\}^{1/2} \\ |u|_{H_{**}^m(\mathbf{R}_+^n)} &= \left\{ \sum_{\substack{|\alpha|+2k \leq m+1 \\ |\alpha|+k \leq m}} |\partial_{\tan}^\alpha \partial_n^k u|_{L^2(\mathbf{R}_+^n)}^2 \right\}^{1/2} \\ |u|_{H_{\tan}^m(\mathbf{R}_+^n)} &= \left\{ \sum_{|\alpha| \leq m} |\partial_{\tan}^\alpha u|_{L^2(\mathbf{R}_+^n)}^2 \right\}^{1/2}. \end{aligned}$$

It is noticed that we may replace the operator  $\partial_{\tan}^\alpha$  with

$$\partial_*^\alpha = x_n^{\alpha_n} \partial_1^{\alpha_1} \cdots \partial_{n-1}^{\alpha_{n-1}} \partial_n^{\alpha_n}$$

to obtain the same definitions of the spaces as Definition 2 and the equivalent norms to the original ones. We often make use of this observation.

Returning to the case of the domain  $\Omega$ , we choose a finite open covering  $\{V_k; 0 \leq k \leq N\}$  of  $\bar{\Omega}$  with the properties

- (1)  $V_0$  is a relatively compact and open subset of  $\Omega$ ;
- (2)  $V_k$ ,  $1 \leq k \leq N$ , is diffeomorphic to an open ball  $B_k$  in  $\mathbf{R}^n$  with center at the origin by a  $C^\infty$ -diffeomorphism  $\Phi_k$  satisfying

$$\Phi_k(V_k \cap \Omega) = B_k \cap \mathbf{R}_+^n, \quad \Phi_k(V_k \cap \Gamma) = B_k \cap \partial \mathbf{R}_+^n;$$

and then a partition of unity  $\{\varphi_k; 0 \leq k \leq N\}$  subordinate to the covering. We cut off a function on  $\Omega$  by  $\varphi_k$  and carry out the change of variables. Since any tangential vector field is represented in the local chart in  $B_k \cap \mathbf{R}_+^n$  by a linear combination of the operators  $\partial_1, \dots, \partial_{n-1}$  and  $x_n \partial_n$  with coefficients in  $C^\infty$ -functions,  $u \in L^2(\Omega)$  belongs to  $H_*^m(\Omega)$  if and only if  $\varphi_0 u \in H^m(\Omega)$  and  $(\varphi_k u) \circ \Phi_k^{-1} \in H_*^m(\mathbf{R}_+^n)$ ,  $1 \leq k \leq N$ .  $H_{**}^m(\Omega)$  and  $H_{\tan}^m(\Omega)$  are characterized similarly by means of  $H_{**}^m(\mathbf{R}_+^n)$  and  $H_{\tan}^m(\mathbf{R}_+^n)$  respectively. Thus,  $H_*^m(\Omega)$ ,  $H_{**}^m(\Omega)$  and  $H_{\tan}^m(\Omega)$  are Hilbert spaces with respective norms

$$|u|_{H_*^m(\Omega)} = \left\{ |\varphi_0 u|_{H^m(\Omega)}^2 + \sum_{k=1}^N |(\varphi_k u) \circ \Phi_k^{-1}|_{H_*^m(\mathbf{R}_+^n)}^2 \right\}^{1/2}$$

$$|u|_{H_{**}^m(\Omega)} = \left\{ |\varphi_0 u|_{H^m(\Omega)}^2 + \sum_{k=1}^N |(\varphi_k u) \circ \Phi_k^{-1}|_{H_{**}^m(\mathbf{R}_+^n)}^2 \right\}^{1/2}$$

$$|u|_{H_{\tan}^m(\Omega)} = \left\{ |\varphi_0 u|_{H^m(\Omega)}^2 + \sum_{k=1}^N |(\varphi_k u) \circ \Phi_k^{-1}|_{H_{\tan}^m(\mathbf{R}_+^n)}^2 \right\}^{1/2}.$$

Let  $C^m(\overline{\Omega})$ ,  $m \in \mathbf{Z}_+$ , be the space of  $m$  times continuously differentiable functions on  $\overline{\Omega}$ . Using  $C^0(\overline{\Omega})$  in place of  $L^2(\Omega)$ , we define the spaces  $C_*^m(\overline{\Omega})$ ,  $C_{**}^m(\overline{\Omega})$  and  $C_{\tan}^m(\overline{\Omega})$  as in Definition 1. The spaces  $C_*^m(\overline{\mathbf{R}_+^n})$ ,  $C_{**}^m(\overline{\mathbf{R}_+^n})$  and  $C_{\tan}^m(\overline{\mathbf{R}_+^n})$  are given as in Definition 2. These spaces are normed in the same way as above and become Banach spaces.

It is well-known that a function in  $H^m(\Omega)$  has the trace on the boundary. The trace belongs to  $H^{m-1/2}(\Gamma)$ . This is also true of a function in  $H_{**}^m(\Omega)$ . Let  $u \in H_{**}^m(\Omega)$ . Writing  $x = (x', x_n)$ ,  $x' \in \mathbf{R}^{n-1}$ ,  $x_n \in \mathbf{R}^1$ , we regard  $(\varphi_k u) \circ \Phi_k^{-1}$  as an element of  $W_2^1(\mathbf{R}_{x_n+}^1; H^{m-1}(\mathbf{R}_{x'}^{n-1})) \cap L^2(\mathbf{R}_{x_n+}^1; H^m(\mathbf{R}_{x'}^{n-1}))$  and apply the trace theorem of Lions (Lions–Magenes [9]). Then, the boundary value  $(\varphi_k u) \circ \Phi_k^{-1}|_{x_n=0}$  exists and lies in

$$\left[ H^{m-1}(\mathbf{R}_{x'}^{n-1}), H^m(\mathbf{R}_{x'}^{n-1}) \right]_{\frac{1}{2}} = H^{m-1/2}(\mathbf{R}_{x'}^{n-1}).$$

Thus, the trace operator  $\gamma_0 : u \mapsto u|_{\Gamma}$  is defined as a linear continuous map from  $H_{**}^m(\Omega)$  to  $H^{m-1/2}(\Gamma)$ . Similarly, when  $m \geq 2$ ,  $u \in H_*^m(\Omega)$  has the trace which belongs to  $H^{m-1}(\Gamma)$ . For several results on the higher order traces and the characterization of the ranges of the trace operators we refer the reader to Ohno–Shizuta–Yanagisawa [14] and Shizuta–Yabuta [22].

We are concerned with solutions of the problem (1.1) some components of which lie in  $H_{**}^m(\Omega)$  while the others in  $H_*^m(\Omega)$  after certain transformation of unknown functions. Such a structure of solutions is known as the extra regularity in the literature [15], [20], [21], [22] and realized in the following function space. If  $\mathbf{L} \in C^\infty(\overline{\Omega})$  vanishes on  $\Gamma$ , we have  $\mathbf{L}u \in H_{**}^m(\Omega)$  for any  $u \in H_*^m(\Omega)$ . Moreover,  $\gamma_0[\mathbf{L}u] = 0$  holds since  $C^\infty(\overline{\Omega})$  is dense in  $H_*^m(\Omega)$ . From this observation the subspace of  $H_*^m(\Omega)$  determined from  $\mathbf{P} \in C^\infty(\overline{\Omega})$  by

$$\{u \in H_*^m(\Omega); \mathbf{P}u \in H_{**}^m(\Omega)\}$$

depends only on the boundary value  $P = \gamma_0[\mathbf{P}]$ . We denote this space by  $\mathcal{H}_P^m(\Omega)$ . This is a Hilbert space with the norm

$$|u|_{\mathcal{H}_P^m(\Omega)} = \left\{ |u|_{H_*^m(\Omega)}^2 + |\mathbf{P}u|_{H_{**}^m(\Omega)}^2 \right\}^{1/2}.$$

For  $u \in \mathcal{H}_P^m(\Omega)$  the trace  $\gamma_0[\mathbf{P}u] \in H^{m-1/2}(\Gamma)$  depends only on  $P$ , which is denoted by  $(P\gamma_0)[u]$ . The boundary condition of the problem (1.1) is described by using

the closed subspace of  $\mathcal{H}_P^m(\Omega)$  given by

$$\widehat{\mathcal{H}}_P^m(\Omega) = \{u \in \mathcal{H}_P^m(\Omega); (P\gamma_0)[u] = 0\}.$$

Finally, we introduce several function spaces on intervals. All the spaces are Banach spaces. Let  $I$  be a finite open interval. We define

$$\begin{aligned} X^m(\bar{I}; \Omega) &= \bigcap_{j=0}^m C^{m-j}(\bar{I}; H^j(\Omega)) \\ Y^m(I; \Omega) &= \bigcap_{j=0}^m W_\infty^{m-j}(I; H^j(\Omega)) \\ W_q^m(I; \Omega) &= \bigcap_{j=0}^m W_q^{m-j}(I; H^j(\Omega)). \end{aligned}$$

In this definition we replace  $H^j(\Omega)$  with  $H_*^j(\Omega)$ ,  $H_{**}^j(\Omega)$  and  $H_{\tan}^j(\Omega)$  and obtain the spaces  $X_*^m(\bar{I}; \Omega)$ ,  $Y_*^m(I; \Omega)$ ,  $W_{q*}^m(I; \Omega)$ ;  $X_{**}^m(\bar{I}; \Omega)$ ,  $Y_{**}^m(I; \Omega)$ ,  $W_{q**}^m(I; \Omega)$  and  $X_{\tan}^m(\bar{I}; \Omega)$ ,  $Y_{\tan}^m(I; \Omega)$ ,  $W_{q\tan}^m(I; \Omega)$  respectively. Corresponding function spaces in the half space  $\mathbf{R}_+^n$  are defined in the same way. For  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^{n+1}$  we denote the differential operator  $\partial_0^{\alpha_0} \partial_1^{\alpha_1} \dots \partial_{n-1}^{\alpha_{n-1}} (x_n \partial_n)^{\alpha_n}$  by  $\partial_{\tan}^\alpha$  and  $x_n^{\alpha_n} \partial_0^{\alpha_0} \partial_1^{\alpha_1} \dots \partial_{n-1}^{\alpha_{n-1}} \partial_n^{\alpha_n}$  by  $\partial_*^\alpha$ . For  $P \in C^\infty(\Gamma)$  we put

$$\mathcal{X}_P^m(\bar{I}; \Omega) = \bigcap_{j=0}^m C^{m-j}(\bar{I}; \mathcal{H}_P^j(\Omega)).$$

### 3. Assumptions and main results.

We state the main results in two theorems. One deals with the existence of solutions of first order regularity. The other is concerned with the higher order regularity of solutions. We make use of the first theorem to show the latter. The statements are given in such a way as they are applied to the problem in which the coefficient matrices lie in the same type of function space as that of solutions, the linearized problem of quasilinear equations kept in mind.

Let  $\Omega$  be a bounded open set in  $\mathbf{R}^n$ ,  $n \geq 2$ , with smooth boundary  $\Gamma$ .  $\nu(x) = {}^t(\nu_1(x), \dots, \nu_n(x))$  denotes the unit outward normal to  $\Gamma$ . Supposing that  $A_j(t, x)$ ,  $0 \leq j \leq n+1$ , and  $Q(x)$  are  $l_0 \times l_0$  matrix-valued functions on  $[0, T] \times \bar{\Omega}$  and  $\Gamma$  respectively, we list the conditions imposed on (1.1).

(H.1).  $A_j(t, x)$ ,  $0 \leq j \leq n$ , are hermitian and  $A_0(t, x)$  is positive definite at each point  $(t, x) \in [0, T] \times \bar{\Omega}$ . There exists a positive constant  $K_0$  such that

$$A_0(t, x) \geq K_0 I, \quad (t, x) \in [0, T] \times \bar{\Omega}.$$

(H.2). The subspace  $\ker Q(x)$  is maximal nonnegative at each point  $(t, x) \in [0, T] \times \Gamma$ , that is, the boundary matrix  $A_\nu(t, x) = \sum_{j=1}^n \nu_j(x) A_j(t, x)$  is nonnegative on

the subspace  $\ker Q(x)$  and any subspace which enjoys this property and contains  $\ker Q(x)$  must coincide with  $\ker Q(x)$ .

(H.3). There exists a function  $P$  on  $\Gamma$  with values in  $l_0 \times l_0$  matrices such that  $\ker A_\nu(t, x) = \ker P(x)$  holds at each point  $(t, x) \in [0, T] \times \Gamma$ . The rank of  $P(x)$  is a constant  $l_1 \in (0, l_0)$  everywhere on  $\Gamma$ .

(H.4). The rank of  $Q(x)$  is a constant  $l_2$  everywhere on  $\Gamma$ .

**Remark 3.1.** As was proved in [8], (H.2) implies

$$(3.1) \quad \ker A_\nu(t, x) \subset \ker Q(x), \quad (t, x) \in [0, T] \times \Gamma.$$

**Remark 3.2.** In the treatment of the equation of ideal magneto-hydrodynamics with a perfectly conducting wall condition under a certain constraint on the initial data the boundary matrix of the linearized equation is determined from the shape of  $\Omega$  only, and does not depend on a particular choice of functions about which the quasilinear equation is linearized (Yanagisawa-Matsumura [29]). Hence, the hypothesis (H.3) and the assumption on the smoothness of  $P$  in the theorems below are not too restrictive in application, though the other types of hypotheses are possible if we confine ourselves to the linear equation (1.1) with smooth coefficients.

**Theorem 1.** Assume that

$$(3.2) \quad \begin{cases} A_j \in W_\infty^1(0, T; C^1(\bar{\Omega})) \cap L^\infty(0, T; C_{**}^2(\bar{\Omega})), & 0 \leq j \leq n, \\ A_{n+1} \in W_\infty^1(0, T; C^0(\bar{\Omega})) \cap L^\infty(0, T; C_*^1(\bar{\Omega})) \end{cases}$$

and  $P, Q \in C^\infty(\Gamma)$ . Then, the problem (1.1) has a unique solution in  $\mathcal{X}_P^1([0, T]; \Omega)$  for  $(f, F) \in (\mathcal{H}_P^1(\Omega) \cap \widehat{\mathcal{H}}_Q^1(\Omega)) \times W_{1*}^1(0, T; \Omega)$ .

**Theorem 2.** Let  $m \geq 2$  and put  $r = \max\{m, 2[n/2] + 6\}$ . We assume that

$$(3.3) \quad A_j \in Y_*^r(0, T; \Omega), \quad 0 \leq j \leq n+1,$$

and  $P, Q \in C^\infty(\Gamma)$ . Suppose that  $u \in \mathcal{X}_P^{m-1}([0, T]; \Omega)$  satisfies (1.1). Then, if  $F$  belongs to  $W_{1*}^m(0, T; \Omega)$  and

$$(3.4) \quad f_p \equiv \partial_t^p u(0) \in \mathcal{H}_P^{m-p}(\Omega) \cap \widehat{\mathcal{H}}_Q^{m-p}(\Omega), \quad 0 \leq p \leq m-1,$$

we have  $u \in \mathcal{X}_P^m([0, T]; \Omega)$ .

It is worthwhile to mention the meaning of the boundary condition in (1.1). Let  $\mathcal{P}(x)$  and  $\mathcal{Q}(x)$  be the orthogonal projections to  $(\ker P(x))^\perp$  and  $(\ker Q(x))^\perp$  respectively. Since  $P(x)$  and  $Q(x)$  are of constant ranks on  $\Gamma$  and dependent on  $x$



smoothly, so are  $\mathcal{P}(x)$  and  $\mathcal{Q}(x)$ . By (3.1) we have  $\ker \mathcal{P}(x) \subset \ker \mathcal{Q}(x)$  and hence  $\mathcal{Q}(x) = \mathcal{Q}(x)\mathcal{P}(x)$ . Therefore,

$$\mathcal{H}_{\mathcal{P}}^m(\Omega) = \mathcal{H}_{\mathcal{P}}^m(\Omega) \subset \mathcal{H}_{\mathcal{Q}\mathcal{P}}^m(\Omega) = \mathcal{H}_{\mathcal{Q}}^m(\Omega) = \mathcal{H}_{\mathcal{Q}}^m(\Omega).$$

This implies  $\mathcal{X}_{\mathcal{P}}^m([0, T]; \Omega) \subset \mathcal{X}_{\mathcal{Q}}^m([0, T]; \Omega)$ . Thus, the condition " $Qu = 0$  on  $[0, T] \times \Gamma$ " for  $u \in \mathcal{X}_{\mathcal{P}}^m([0, T]; \Omega)$  makes sense by saying  $u(t) \in \widehat{\mathcal{H}}_{\mathcal{Q}}^m(\Omega)$ ,  $0 \leq t \leq T$ . By the continuity of the trace operator  $Q\gamma_0$  it is also proved that a function  $u \in \mathcal{X}_{\mathcal{P}}^m([0, T]; \Omega)$  with the boundary condition must satisfy (3.4).

We may express  $f_p$  in Theorem 2 as a linear combination of the derivatives of  $f$  and the values at  $t = 0$  of the derivatives of  $F$  with coefficients in  $l_0 \times l_0$  matrix-valued functions on  $\Omega$ . The relations between  $f$  and  $F$  given by (3.4) is called the compatibility condition of order  $m - 1$ . When  $m = 1$ , the compatibility condition is stated that  $f$  belongs to  $\mathcal{H}_P^1(\Omega) \cap \widehat{\mathcal{H}}_Q^1(\Omega)$ . Shizuta-Yabuta [22] showed that if a function  $u \in X_*^m([0, T]; \Omega)$  satisfies the first equation in (1.1) with  $F \in W_{1*}^m(0, T; \Omega)$ , it necessarily belongs to  $\mathcal{X}_{\mathcal{P}}^m([0, T]; \Omega)$ . Hence to solve the problem (1.1) in the class  $X_*^m([0, T]; \Omega)$  we must impose the compatibility condition on the data. The above theorems say that we can solve the problem (1.1) in the class  $\mathcal{X}_{\mathcal{P}}^m([0, T]; \Omega)$  for any data satisfying the compatibility condition.

In this paper, instead of proving the theorems themselves, we will present the ideas of the proofs using an equation with smooth coefficients in the half space. Let us consider the problem (1.1) in the half space  $\mathbf{R}_+^n$ . All the hypotheses (H.1) to (H.4) are meaningful also in the case  $\Omega = \mathbf{R}_+^n$ . We write

$$A_j = \begin{pmatrix} A_j^{11} & A_j^{12} \\ A_j^{21} & A_j^{22} \end{pmatrix}$$

with  $A_j^{11}$  and  $A_j^{22}$ , square matrices of order  $l_1$  and  $l_0 - l_1$  respectively and  $A_j^{12} = (A_j^{21})^*$ , an  $l_1 \times (l_0 - l_1)$  matrix. In addition to the hypotheses above the boundary matrix  $-A_n$  is assumed to have the properties

- (1)  $A_n^{11}$  is not singular on  $[0, T] \times \partial\mathbf{R}_+^n$ ;
- (2)  $A_n^{12} = (A_n^{21})^*$  and  $A_n^{22}$  vanish on  $[0, T] \times \partial\mathbf{R}_+^n$ .

We further assume that there exists a positive constant  $c_0$  such that

$$(3.5) \quad (A_n^{11})^* A_n^{11} \geq c_0^2 I, \quad [0, T] \times \overline{\mathbf{R}_+^n}.$$

The matrices  $P$  and  $Q$  are assumed to be of the forms

$$P = \begin{pmatrix} E_{l_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} E_{l_2} & 0 \\ 0 & 0 \end{pmatrix},$$

where  $E_l$  is the identity matrix of order  $l$ . The relation (3.1) implies  $l_1 \geq l_2$ .

As for the smoothness of the coefficients we put

$$(3.6) \quad A_j \in \tilde{B}^\infty([0, T] \times \overline{\mathbf{R}_+^n}), \quad 0 \leq j \leq n+1,$$

in place of (3.2) and (3.3), where  $\tilde{B}^m([0, T] \times \overline{\mathbf{R}_+^n})$  is the space of functions on  $[0, T] \times \overline{\mathbf{R}_+^n}$  whose derivatives with respect to the operators  $\partial_0, \dots, \partial_n$  and  $x_n \partial_n$  of order up to  $m$  are bounded and continuous on  $[0, T] \times \overline{\mathbf{R}_+^n}$ . We set

$$\begin{aligned} \mathcal{H}_P^m(\mathbf{R}_+^n) &= \{u \in H_*^m(\mathbf{R}_+^n); Pu \in H_{**}^m(\mathbf{R}_+^n)\} \\ \widehat{\mathcal{H}}_Q^m(\mathbf{R}_+^n) &= \{u \in H_*^m(\mathbf{R}_+^n); Qu \in H_{**}^m(\mathbf{R}_+^n), \gamma_0[Qu] = 0\} \\ \mathcal{X}_P^m([0, T]; \mathbf{R}_+^n) &= \bigcap_{j=0}^m C^{m-j}([0, T]; \mathcal{H}_P^j(\mathbf{R}_+^n)). \end{aligned}$$

Then, all the statements in the theorems on the equation in  $\Omega = \mathbf{R}_+^n$  make sense. In the sequel we write  $u \in C^{l_0}$  as  ${}^t(u_I, u_{II})$  with  $u_I \in C^{l_1}$  and  $u_{II} \in C^{l_0-l_1}$ . For the sake of simplicity we assume that the support of the data  $(f, F)$  is compact, and so is the support of the solution by the finiteness of the speed of the propagation.

#### 4. Existence of solutions of first order regularity.

We solve the problem (1.1) by the method of non-characteristic regularization. Let  $\eta$  be a positive parameter. We consider the approximating problem to (1.1):

$$(1.1_\eta) \quad \begin{cases} \sum_{j=0}^n A_j \partial_j u - \eta \partial_n u + A_{n+1} u = F & \text{in } [0, T] \times \mathbf{R}_+^n \\ Qu = 0 & \text{on } [0, T] \times \partial \mathbf{R}_+^n \\ u(0) = f & \text{on } \mathbf{R}_+^n. \end{cases}$$

The boundary matrix to the problem (1.1<sub>η</sub>) is  $A_\nu^\eta(t, x) \equiv -A_n(t, x) + \eta I$ . As was proved by Schochet [19],  $A_\nu^\eta(t, x)$  is regular and the subspace  $\ker Q$  is maximal nonnegative at each point  $(t, x) \in [0, T] \times \partial \mathbf{R}_+^n$  if  $\eta$  is small enough. Hence the problem (1.1<sub>η</sub>) satisfies all the hypotheses in Theorem 1 but (H.3), which is replaced by the hypothesis that the boundary matrix has full rank everywhere on the lateral boundary. For such a problem the existence of solutions in the class  $X^1([0, T]; \mathbf{R}_+^n)$  is known. See Rauch–Massey III [18]. Making use of this fact, and the data  $(f, F)$  fixed in the space  $H^1(\mathbf{R}_+^n) \times W_{1*}^1(0, T; \mathbf{R}_+^n)$ , we first prove that the sequence of solutions to (1.1<sub>η</sub>) remains bounded in  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$  as  $\eta$  tends to 0. Next, by a sort of weak compactness method we find a solution to (1.1) in  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ . Finally, by approximating the data the existence theorem in the general case is established. The uniqueness of solutions in the class  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$  follows from the standard energy estimate.

**The first step.** Suppose that the data  $(f, F) \in H^1(\mathbf{R}_+^n) \times W_{1*}^1(0, T; \mathbf{R}_+^n)$  satisfies  $Q\gamma_0[f] = 0$ . If  $\eta > 0$  is small enough, (1.1<sub>η</sub>) has a unique solution in  $X^1([0, T]; \mathbf{R}_+^n)$ . Let us derive some uniform estimates of  $\partial_*^\alpha u$ ,  $\alpha \in \mathbf{Z}_+^{n+1}$ ,  $|\alpha| \leq 1$ , and  $\partial_n u_I$  with respect to the parameter  $\eta$ .

We first consider the case  $\alpha = 0$ . By the hypothesis (H.1) the energy equality

$$\begin{aligned} & \partial_t (A_0(t)u(t), u(t))_{L^2(\mathbf{R}_+^n)} \\ & + \left( (A_{n+1}(t) + A_{n+1}(t)^* - \sum_{j=0}^n \partial_j A_j(t))u(t), u(t) \right)_{L^2(\mathbf{R}_+^n)} \\ & - (A_n(t)u(t), u(t))_{L^2(\partial\mathbf{R}_+^n)} + \eta(u(t), u(t))_{L^2(\partial\mathbf{R}_+^n)} \\ & = 2\Re(u(t), F(t))_{L^2(\mathbf{R}_+^n)} \end{aligned}$$

holds. Since  $-A_n$  is nonnegative on  $\ker Q$ , we have

$$e^{\lambda_0 t} |A_0(t)^{1/2} u(t)|_{L^2(\mathbf{R}_+^n)} \leq |A_0(0)^{1/2} u(0)|_{L^2(\mathbf{R}_+^n)} + \int_0^t e^{\lambda_0 s} |A_0(s)^{-1/2} F(s)|_{L^2(\mathbf{R}_+^n)} ds$$

with a constant  $\lambda_0$  satisfying

$$\frac{1}{2} A_0(t)^{-1/2} \left( A_{n+1}(t) + A_{n+1}(t)^* - \sum_{j=0}^n \partial_j A_j(t) \right) A_0(t)^{-1/2} \geq \lambda_0 I.$$

Henceforth we often make use of similar arguments to estimate solutions of various symmetric systems.

In order to estimate  $\partial_*^\alpha u$ ,  $|\alpha| = 1$ , we use the mollifier  $\mathcal{M}_\varepsilon$  in Appendix A. Choose  $\varepsilon_0 \in (0, T)$ . For  $\alpha \in \mathbf{Z}_+^{n+1}$ ,  $|\alpha| \leq 1$ ,  $0 < \varepsilon < \varepsilon_0$ , we put

$$u_\varepsilon^\alpha = \partial_*^\alpha (\mathcal{M}_\varepsilon u).$$

$u_\varepsilon^\alpha$ ,  $|\alpha| = 1$ , belongs to  $X^1([0, T - \varepsilon_0]; \mathbf{R}_+^n)$  and satisfies the equation

$$\begin{cases} \sum_{j=0}^n A_j \partial_j u_\varepsilon^\alpha + A_{n+1} u_\varepsilon^\alpha - \eta \partial_*^\alpha \partial_n \mathcal{M}_\varepsilon u = J_\varepsilon^\alpha & \text{in } [0, T - \varepsilon_0] \times \mathbf{R}_+^n \\ Qu_\varepsilon^\alpha = 0 & \text{on } [0, T - \varepsilon_0] \times \partial\mathbf{R}_+^n. \end{cases}$$

The forcing term  $J_\varepsilon^\alpha$  is expressed as  $J_\varepsilon^\alpha = J^\alpha(u_\varepsilon^0, \mathcal{F}_\varepsilon)$ , where

$$J^\alpha(v, G) = \alpha_n A_n \partial_n v - \sum_{j=0}^n \partial_*^\alpha A_j \partial_j v - \partial_*^\alpha A_{n+1} v + \partial_*^\alpha G$$

and

$$\mathcal{F}_\varepsilon = \sum_{j=0}^n [A_j \partial_j, \mathcal{M}_\varepsilon] u - \eta [\partial_n, \mathcal{M}_\varepsilon] u + [A_{n+1}, \mathcal{M}_\varepsilon] u + \mathcal{M}_\varepsilon F.$$

We derive the estimate of  $u_\varepsilon^\alpha$  as above and let  $\varepsilon \rightarrow 0$ . Since  $u \in X^1([0, T]; \mathbf{R}_+^n)$ , we have  $\mathcal{M}_\varepsilon u \rightarrow u$  in  $X^1([0, T - \varepsilon_0]; \mathbf{R}_+^n)$ . By Lemma A.1 the commutators

$[A_j \partial_j, \mathcal{M}_\varepsilon]u$ ,  $0 \leq j \leq n$ , and  $[\partial_n, \mathcal{M}_\varepsilon]u$  tend to 0 in  $W_{1*}^1(0, T - \varepsilon_0; \mathbf{R}_+^n)$ . Hence,  $\{\mathcal{F}_\varepsilon\}$  converges to  $F$  in  $W_{1*}^1(0, T - \varepsilon_0; \mathbf{R}_+^n)$ . Consequently, we obtain

$$\begin{aligned} e^{\lambda_0 t} |A_0(t)^{1/2} \partial_*^\alpha u(t)|_{L^2(\mathbf{R}_+^n)} \\ \leq |A_0(0)^{1/2} \partial_*^\alpha u(0)|_{L^2(\mathbf{R}_+^n)} + \int_0^t e^{\lambda_0 s} |A_0(s)^{-1/2} J^\alpha(u, F)(s)|_{L^2(\mathbf{R}_+^n)} ds. \end{aligned}$$

We have

$$\begin{aligned} (4.1) \quad & |A_0(s)^{-1/2} J^\alpha(u, F)(s)|_{L^2(\mathbf{R}_+^n)} \\ & \leq K_0^{-1/2} \left\{ \left( |\partial_*^\alpha A_n^{11}|_{L^\infty} + |\partial_*^\alpha A_n^{21}|_{L^\infty} \right) |\partial_n u_I(s)|_{L^2(\mathbf{R}_+^n)} \right. \\ & \quad + \alpha_n \left( |A_n^{11}|_{L^\infty} + |A_n^{21}|_{L^\infty} \right) |\partial_n u_I(s)|_{L^2(\mathbf{R}_+^n)} \\ & \quad + \left( |x_n^{-1} \partial_*^\alpha A_n^{12}|_{L^\infty} + |x_n^{-1} \partial_*^\alpha A_n^{22}|_{L^\infty} \right) |x_n \partial_n u_{II}(s)|_{L^2(\mathbf{R}_+^n)} \\ & \quad + \alpha_n \left( |x_n^{-1} A_n^{12}|_{L^\infty} + |x_n^{-1} A_n^{22}|_{L^\infty} \right) |\partial_*^\alpha u_{II}(s)|_{L^2(\mathbf{R}_+^n)} \\ & \quad + \sum_{j=0}^{n-1} |\partial_*^\alpha A_j|_{L^\infty} |\partial_j u(s)|_{L^2(\mathbf{R}_+^n)} + |\partial_*^\alpha A_{n+1}|_{L^\infty} |u(s)|_{L^2(\mathbf{R}_+^n)} \Big\} \\ & \quad + |A_0(s)^{-1/2} \partial_*^\alpha F(s)|_{L^2(\mathbf{R}_+^n)}. \end{aligned}$$

To estimate the norm of  $\partial_n u_I$  on the right-hand side of (4.1) we use the equation

$$A_n^{11} \partial_n u_I = \eta \partial_n u_I - \sum_{j=0}^{n-1} A_j^{11} \partial_j u_I - \sum_{j=0}^n A_j^{12} \partial_j u_{II} - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II} + F_I.$$

together with (3.5) to obtain

$$\begin{aligned} & (c_0 - \eta) |\partial_n u_I(s)|_{L^2(\mathbf{R}_+^n)} \\ & \leq \sum_{j=0}^{n-1} |A_j^{11}|_{L^\infty} |\partial_j u_I(s)|_{L^2(\mathbf{R}_+^n)} \\ & \quad + \sum_{j=0}^{n-1} |A_j^{12}|_{L^\infty} |\partial_j u_{II}(s)|_{L^2(\mathbf{R}_+^n)} + |x_n^{-1} A_n^{12}|_{L^\infty} |x_n \partial_n u_{II}(s)|_{L^2(\mathbf{R}_+^n)} \\ & \quad + |A_{n+1}^{11}|_{L^\infty} |u_I(s)|_{L^2(\mathbf{R}_+^n)} + |A_{n+1}^{12}|_{L^\infty} |u_{II}(s)|_{L^2(\mathbf{R}_+^n)} + |F_I(s)|_{L^2(\mathbf{R}_+^n)}. \end{aligned}$$

Combining these estimates, then summing up those of  $\partial_*^\alpha u$  for  $|\alpha| \leq 1$ , we get

$$\begin{aligned} & e^{\lambda_0 t} \sum_{|\alpha| \leq 1} |A_0(t)^{1/2} \partial_*^\alpha u(t)|_{L^2(\mathbf{R}_+^n)} \\ & \leq \sum_{|\alpha| \leq 1} |A_0(0)^{1/2} \partial_*^\alpha u(0)|_{L^2(\mathbf{R}_+^n)} \end{aligned}$$

$$\begin{aligned}
& + MK_0^{-1} \int_0^t e^{\lambda_0 s} \sum_{|\alpha| \leq 1} |A_0(s)^{1/2} \partial_*^\alpha u(s)|_{L^2(\mathbf{R}_+^n)} ds \\
& + M' \int_0^t e^{\lambda_0 s} \sum_{|\alpha| \leq 1} |A_0(s)^{-1/2} \partial_*^\alpha F(s)|_{L^2(\mathbf{R}_+^n)} ds
\end{aligned}$$

with constants  $M$  and  $M'$  independent of  $\eta$ . Putting

$$E(t) = \sum_{|\alpha| \leq 1} |A_0(t)^{1/2} \partial_*^\alpha u(t)|_{L^2(\mathbf{R}_+^n)}, \quad F(t) = \sum_{|\alpha| \leq 1} |A_0(t)^{-1/2} \partial_*^\alpha F(t)|_{L^2(\mathbf{R}_+^n)},$$

we obtain by Gronwall's inequality that

$$(4.2) \quad E(t) \leq E(0) \exp(-\lambda_1 t) + M' \int_0^t \exp(-\lambda_1(t-s)) F(s) ds$$

with  $\lambda_1 = \lambda_0 - M/K_0$ . We have also

$$(4.3) \quad |\partial_n u_I(t)|_{L^2(\mathbf{R}_+^n)} \leq M'' \left\{ \sum_{|\alpha| \leq 1} |\partial_*^\alpha u(t)|_{L^2(\mathbf{R}_+^n)} + |F(t)|_{L^2(\mathbf{R}_+^n)} \right\}$$

with a constant  $M''$  independent of  $\eta$ .

**The second step.** Let  $u_\eta$  be the solution of (1.1) $_\eta$  in  $X^1([0, T]; \mathbf{R}_+^n)$ . Since

$$\partial_t u_\eta(0) = A_0(0)^{-1} \left\{ F(0) - \sum_{j=1}^n A_j(0) \partial_j f + \eta \partial_n f - A_{n+1}(0) f \right\},$$

$\{\partial_t u_\eta(0)\}$  converges in  $L^2(\mathbf{R}_+^n)$  as  $\eta$  tends to 0. Hence, from the estimates (4.2), (4.3) the sequence  $\{u_\eta\}$  is bounded in  $W_\infty^1(0, T; L^2(\mathbf{R}_+^n)) \cap L^\infty(0, T; \mathcal{H}_P^1(\mathbf{R}_+^n) \cap \widehat{\mathcal{H}}_Q^1(\mathbf{R}_+^n))$ . We apply Lemma B in Appendix to  $\{u_\eta\}$  and find a subsequence  $\{u_{\eta_j}\}$  and  $u \in W_\infty^1(0, T; L^2(\mathbf{R}_+^n)) \cap L^\infty(0, T; \mathcal{H}_P^1(\mathbf{R}_+^n) \cap \widehat{\mathcal{H}}_Q^1(\mathbf{R}_+^n))$  such that

$$\lim_{j \rightarrow \infty} u_{\eta_j}(t) = u(t) \quad \text{weakly in } \mathcal{H}_P^1(\mathbf{R}_+^n) \cap \widehat{\mathcal{H}}_Q^1(\mathbf{R}_+^n).$$

The convergence is uniform with respect to  $t \in [0, T]$  and  $u(0) = f$  holds.

$u$  is a solution of (1.1) in  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ . To show this we rely on some basic facts in functional analysis. Let  $E$  and  $F$  be normed spaces.  $\mathcal{L}(E, F)$  denotes the space of bounded linear operators from  $E$  to  $F$ . We write  $\mathcal{L}(E, E) = \mathcal{L}(E)$ . We define the linear operators  $\mathcal{A}_0(t)$  and  $\mathcal{L}(t)$ ,  $0 \leq t \leq T$ , by

$$\begin{aligned}
(\mathcal{A}_0(t)g)(x) &= A_0(t, x)g(x) \\
(\mathcal{L}(t)g)(x) &= \sum_{j=1}^n A_j(t, x) \partial_j g(x) + A_{n+1}(t, x)g(x).
\end{aligned}$$

Obviously,  $\mathcal{A}_0(t)$  belongs to  $\mathcal{L}(L^2(\mathbf{R}_+^n))$  with bounded inverse and  $\mathcal{A}_0(\cdot), \mathcal{A}_0(\cdot)^{-1} \in C^0([0, T]; \mathcal{L}(L^2(\mathbf{R}_+^n)))$ . We express  $\mathcal{L}(t)g$  as

$$\sum_{j=1}^{n-1} A_j \partial_j g + A_n(I - P) \partial_n g + A_n \partial_n(Pg) + A_{n+1}g$$

and notice that the operator  $\sum_{j=1}^{n-1} A_j \partial_j + A_n(I - P) \partial_n$  is tangential. Then we have  $\mathcal{L}(t) \in \mathcal{L}(\mathcal{H}_P^1(\mathbf{R}_+^n), L^2(\mathbf{R}_+^n))$  and  $\mathcal{L}(\cdot) \in C^0([0, T]; \mathcal{L}(\mathcal{H}_P^1(\mathbf{R}_+^n), L^2(\mathbf{R}_+^n)))$ .

We shall prove  $u \in C_w^1([0, T]; L^2(\mathbf{R}_+^n)) \cap C_w^0([0, T]; \mathcal{H}_P^1(\mathbf{R}_+^n) \cap \widehat{\mathcal{H}}_Q^1(\mathbf{R}_+^n))$  and

$$(4.4) \quad \mathcal{A}_0(t) \partial_t u(t) + \mathcal{L}(t)u(t) = F(t) \quad \text{in } L^2(\mathbf{R}_+^n), \quad 0 \leq t \leq T.$$

*Proof:* Let  $\tilde{\Omega}$  be a relatively compact and open subset of  $\mathbf{R}_+^n$ . For a function  $g$  on  $\mathbf{R}_+^n$  the restriction of  $g$  onto  $\tilde{\Omega}$  is denoted by  $\mathcal{R}g$ . We have  $\mathcal{R} \in \mathcal{L}(L^2(\mathbf{R}_+^n), L^2(\tilde{\Omega})) \cap \mathcal{L}(\mathcal{H}_P^1(\mathbf{R}_+^n), H^1(\tilde{\Omega}))$ . We define the operators  $\tilde{\mathcal{A}}_0(t) \in \mathcal{L}(L^2(\tilde{\Omega}))$ ,  $0 \leq t \leq T$ , by

$$(\tilde{\mathcal{A}}_0(t)g)(x) = A_0(t, x)g(x).$$

$\tilde{\mathcal{A}}_0(t)$  is invertible and  $\tilde{\mathcal{A}}_0(\cdot), \tilde{\mathcal{A}}_0(\cdot)^{-1} \in C^0([0, T]; \mathcal{L}(L^2(\tilde{\Omega})))$ . We see  $\partial_n \in \mathcal{L}(H^1(\tilde{\Omega}), L^2(\tilde{\Omega}))$ . From the equation (1.1 <sub>$\eta$</sub> ) we have

$$\mathcal{R} \partial_t u_{\eta_j}(t) = \mathcal{R} \mathcal{A}_0(t)^{-1} (F(t) - \mathcal{L}(t)u_{\eta_j}(t)) + \eta_j \tilde{\mathcal{A}}_0^{-1}(t) \partial_n \mathcal{R} u_{\eta_j}(t).$$

The right-hand side converges to  $\mathcal{R} \mathcal{A}_0(t)^{-1} (F(t) - \mathcal{L}(t)u(t))$  weakly in  $L^2(\tilde{\Omega})$  uniformly on  $[0, T]$ . Taking the weak limits of the both sides of

$$\mathcal{R}(u_{\eta_j}(t) - f) = \int_0^t \mathcal{R} \partial_t u_{\eta_j}(\tau) d\tau,$$

we obtain

$$\mathcal{R}(u(t) - f) = \int_0^t \mathcal{R} \mathcal{A}_0(\tau)^{-1} (F(\tau) - \mathcal{L}(\tau)u(\tau)) d\tau$$

and immediately

$$\mathcal{R} \left\{ u(t) - f - \int_0^t \mathcal{A}_0(\tau)^{-1} (F(\tau) - \mathcal{L}(\tau)u(\tau)) d\tau \right\} = 0.$$

Since  $\tilde{\Omega}$  is arbitrary, we get

$$u(t) - f - \int_0^t \mathcal{A}_0(\tau)^{-1} (F(\tau) - \mathcal{L}(\tau)u(\tau)) d\tau = 0.$$

This shows that  $u \in C_w^1([0, T]; L^2(\mathbf{R}_+^n))$  and (4.4) holds. ■

We can prove that  $u$  lies in  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$  by using the mollifier  $\mathcal{M}_\varepsilon$ . The detail of the proof will be given in [23].

**The third step.** (1.1) has a unique solution  $u \in \mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$  for  $(f, F) \in H^1(\mathbf{R}_+^n) \times W_{1*}^1(0, T; \mathbf{R}_+^n)$  with  $Q\gamma_0[f] = 0$ . The estimates (4.2) and (4.3) are valid. Since  $\partial_t u(0) = A_0(0)^{-1}(F(0) - \mathcal{L}(0)f)$ , the existence theorem in the general case is proved by approximating  $f \in \mathcal{H}_P^1(\mathbf{R}_+^n) \cap \widehat{\mathcal{H}}_Q^1(\mathbf{R}_+^n)$  by a sequence  $\{f_\varepsilon; \varepsilon > 0\}$  in  $H^1(\mathbf{R}_+^n)$  with  $Q\gamma_0[f_\varepsilon] = 0$ . Let  $S_\varepsilon$  be the shift operator:  $u(x', x_n) \mapsto u(x', x_n + \varepsilon)$ . It is easy to see that  $f_\varepsilon = Pf + (I - P)S_\varepsilon f$  gives a desired sequence in  $H^1(\mathbf{R}_+^n)$ .

### 5. Tangential regularity.

We proceed with the proof of Theorem 2. In this section we show the tangential regularity of order  $m$  of solutions. Let  $m \geq 2$ . Suppose that  $u \in \mathcal{X}_P^{m-1}([0, T]; \mathbf{R}_+^n)$  is a solution of (1.1) with  $F \in W_{1*}^m(0, T; \mathbf{R}_+^n)$  and (3.4). For  $\alpha \in \mathbf{Z}_+^{n+1}$ ,  $|\alpha| \leq m-1$ , we put

$$u^\alpha = \partial_*^\alpha u$$

By the assumption it is clear that  $u^\alpha \in C^0([0, T]; L^2(\mathbf{R}_+^n))$ . We will show that  $u^\alpha$ ,  $|\alpha| = m-1$ , belongs to  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ .

We first prove that  $u^\alpha$  is the strong solution to the equation

$$(5.1) \quad \begin{cases} \sum_{j=0}^n A_j \partial_j u^\alpha + A_{n+1} u^\alpha = J^\alpha & \text{in } [0, T] \times \mathbf{R}_+^n \\ Qu^\alpha = 0 & \text{on } [0, T] \times \partial \mathbf{R}_+^n \\ u^\alpha(0) = u^\alpha(0) & \text{on } \mathbf{R}_+^n \end{cases}$$

with the forcing term  $J^\alpha$  given below in (5.3). Next, choosing suitable functions  $B^{\alpha\beta}$ ,  $\beta \in \mathbf{Z}_+^{n+1}$ ,  $|\beta| = m-1$ , and  $G^\alpha$  on  $[0, T] \times \mathbf{R}_+^n$  with values in square matrices of order  $l_0$  and  $C^{l_0}$  respectively, we show that  $J^\alpha$  is of the form

$$J^\alpha = \sum_{|\beta|=m-1} B^{\alpha\beta} u^\beta + G^\alpha.$$

By Theorem 1 the first order system for the unknown  $(v^\alpha; |\alpha| = m-1)$

$$(5.2) \quad \begin{cases} \sum_{j=0}^n A_j \partial_j v^\alpha + A_{n+1} v^\alpha = \sum_{|\beta|=m-1} B^{\alpha\beta} v^\beta + G^\alpha & \text{in } [0, T] \times \mathbf{R}_+^n \\ Qv^\alpha = 0 & \text{on } [0, T] \times \partial \mathbf{R}_+^n \\ v^\alpha(0) = u^\alpha(0) & \text{on } \mathbf{R}_+^n \end{cases}$$

has a unique solution in the class  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ . This together with the energy estimate for the difference  $u^\alpha - v^\alpha$  leads to the conclusion  $u^\alpha \in \mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ . In the sequel we let  $e_j = (\delta_{jk}) \in \mathbf{Z}_+^{n+1}$ , where  $\delta_{jk}$  is Kronecker's symbol.

**The first step.** Let  $\mathcal{M}_\varepsilon$  be the mollifier in Appendix A. Choosing  $\varepsilon_0 \in (0, T)$ , we define for  $\alpha \in \mathbf{Z}_+^{n+1}$ ,  $|\alpha| \leq m-1$ ,  $0 < \varepsilon < \varepsilon_0$ ,

$$u_\varepsilon^\alpha = \partial_*^\alpha (\mathcal{M}_\varepsilon u).$$

Then,  $u_\varepsilon^\alpha$ ,  $|\alpha| = m-1$ , belongs to  $\mathcal{X}_P^{m-1}([0, T-\varepsilon_0]; \mathbf{R}_+^n)$  and satisfies the equation

$$\begin{cases} \sum_{j=0}^n A_j \partial_j u_\varepsilon^\alpha + A_{n+1} u_\varepsilon^\alpha = J_\varepsilon^\alpha & \text{in } [0, T-\varepsilon_0] \times \mathbf{R}_+^n \\ Qu_\varepsilon^\alpha = 0 & \text{on } [0, T-\varepsilon_0] \times \partial \mathbf{R}_+^n \end{cases}$$

with the forcing term given by  $J_\varepsilon^\alpha = J^\alpha(u_\varepsilon^0, \mathcal{F}_\varepsilon)$ , where

$$J^\alpha(v, G) = \alpha_n A_n \partial_*^{\alpha-e_n} \partial_n v + \sum_{j=0}^n [A_j, \partial_*^\alpha] \partial_j v + [A_{n+1}, \partial_*^\alpha] v + \partial_*^\alpha G$$

and

$$\mathcal{F}_\varepsilon = \sum_{j=0}^n [A_j \partial_j, \mathcal{M}_\varepsilon] u + [A_{n+1}, \mathcal{M}_\varepsilon] u + \mathcal{M}_\varepsilon F.$$

It is clear that  $u_\varepsilon^\alpha$  converges to  $u^\alpha$  in  $C^0([0, T-\varepsilon_0]; \mathbf{R}_+^n)$  as  $\varepsilon \rightarrow 0$ . Putting

$$(5.3) \quad J^\alpha = J^\alpha(u, F),$$

we shall prove that  $u^\alpha$  satisfies the equation (5.1) in the strong sense:

$$(5.4) \quad \lim_{\varepsilon \rightarrow 0} J_\varepsilon^\alpha = J^\alpha \quad \text{in } L^1(0, T-\varepsilon_0; L^2(\mathbf{R}_+^n)).$$

*Proof:* For  $(v, G) \in W_{1*}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n) \times W_{1*}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)$  with  $v_I \in W_{1**}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)$  we have

$$\begin{aligned} & |J^\alpha(v, G)|_{L^1(0, T-\varepsilon_0; L^2(\mathbf{R}_+^n))} \\ & \leq \alpha_n \left( |A_n^{11}|_{L^\infty} + |A_n^{21}|_{L^\infty} \right) |\partial_*^{\alpha-e_n} \partial_n v_I|_{L^1(0, T-\varepsilon_0; L^2(\mathbf{R}_+^n))} \\ & \quad + \alpha_n \left( |x_n^{-1} A_n^{12}|_{L^\infty} + |x_n^{-1} A_n^{22}|_{L^\infty} \right) |\partial_*^\alpha v_{II}|_{L^1(0, T-\varepsilon_0; L^2(\mathbf{R}_+^n))} \\ & \quad + C \sum_{j=0}^{n-1} |A_j|_{\tilde{B}^{m-1}([0, T-\varepsilon_0] \times \overline{\mathbf{R}_+^n})} |\partial_j v|_{W_{1*}^{m-2}(0, T-\varepsilon_0; \mathbf{R}_+^n)} \\ & \quad + C \left( |A_n^{11}|_{\tilde{B}^{m-1}([0, T-\varepsilon_0] \times \overline{\mathbf{R}_+^n})} + |A_n^{21}|_{\tilde{B}^{m-1}([0, T-\varepsilon_0] \times \overline{\mathbf{R}_+^n})} \right) |\partial_n v_I|_{W_{1*}^{m-2}(0, T-\varepsilon_0; \mathbf{R}_+^n)} \\ & \quad + C \left( |A_n^{12}|_{\tilde{B}^{(m-1) \vee 2}([0, T-\varepsilon_0] \times \overline{\mathbf{R}_+^n})} + |A_n^{22}|_{\tilde{B}^{(m-1) \vee 2}([0, T-\varepsilon_0] \times \overline{\mathbf{R}_+^n})} \right) |v_{II}|_{W_{1*}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)} \\ & \quad + C |A_{n+1}|_{\tilde{B}^{m-1}([0, T-\varepsilon_0] \times \overline{\mathbf{R}_+^n})} |v|_{W_{1*}^{m-2}(0, T-\varepsilon_0; \mathbf{R}_+^n)} + |\partial_*^\alpha G|_{L^1(0, T-\varepsilon_0; L^2(\mathbf{R}_+^n))}. \end{aligned}$$

We see  $\mathcal{M}_\varepsilon u_I \rightarrow u_I$  in  $W_{1**}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)$ ,  $\mathcal{M}_\varepsilon u_{II} \rightarrow u_{II}$  in  $W_{1*}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)$  as  $\varepsilon \rightarrow 0$ . The commutators  $[A_j \partial_j, \mathcal{M}_\varepsilon] u$ ,  $0 \leq j \leq n-1$ ,  $[A_n^{l1} \partial_n, \mathcal{M}_\varepsilon] u_I$ ,  $l = 1, 2$ , and  $[A_n^{l2} \partial_n, \mathcal{M}_\varepsilon] u_{II}$ ,  $l = 1, 2$ , tend to 0 in  $W_{1 \tan}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)$  by Lemma A.1 (1), (2) and Lemma A.2 respectively. Hence,  $\mathcal{F}_\varepsilon \rightarrow F$  in  $W_{1 \tan}^{m-1}(0, T-\varepsilon_0; \mathbf{R}_+^n)$ . Combining these with the estimate of  $J^\alpha(v, G)$ , we obtain (5.4). ■



**The second step.** We shall derive the following expression of  $J^\alpha$ :

$$(5.5) \quad J^\alpha = \sum_{|\beta|=m-1} B^{\alpha\beta} u^\beta + G^\alpha,$$

where  $B^{\alpha\beta}$  are functions in  $\tilde{B}^\infty([0, T] \times \overline{\mathbf{R}_+^n})$  taking the values in square matrices of order  $l_0$  and determined from  $A_j$ ,  $0 \leq j \leq n$ , and  $G^\alpha$  is a  $\mathbf{C}^{l_0}$ -valued function in  $W_{1*}^1(0, T; \mathbf{R}_+^n)$  determined from  $u$  and  $F$ .

To begin with we recall the definition (5.3) of  $J^\alpha$ :

$$J^\alpha = \alpha_n A_n \partial_*^{\alpha-e_n} \partial_n u + \sum_{j=0}^n [A_j, \partial_*^\alpha] \partial_j u + [A_{n+1}, \partial_*^\alpha] u + \partial_*^\alpha F.$$

In the first term of  $J^\alpha$  we rewrite the normal derivative  $\partial_n u_I$  by using the equation

$$(5.6) \quad A_n^{11} \partial_n u_I = - \sum_{j=0}^{n-1} A_j^{11} \partial_j u_I - \sum_{j=0}^n A_j^{12} \partial_j u_{II} - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II} + F_I.$$

Then,  $A_n \partial_*^{\alpha-e_n} \partial_n u$  is written as

$$(5.7) \quad - \sum_{j=0}^{n-1} \begin{pmatrix} A_j^{11} & A_j^{12} \\ 0 & 0 \end{pmatrix} u^{\alpha-e_n+e_j} + x_n^{-1} \begin{pmatrix} 0 & 0 \\ A_n^{21} & A_n^{22} \end{pmatrix} u^\alpha + \begin{pmatrix} I^\alpha \\ 0 \end{pmatrix}$$

with

$$I^\alpha = \sum_{j=0}^{n-1} A_n^{11} [(A_n^{11})^{-1} A_j^{11}, \partial_*^{\alpha-e_n}] \partial_j u_I + \sum_{j=0}^n A_n^{11} [(A_n^{11})^{-1} A_j^{12}, \partial_*^{\alpha-e_n}] \partial_j u_{II} \\ + A_n^{11} \partial_*^{\alpha-e_n} \left\{ (A_n^{11})^{-1} (F_I - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II}) \right\}.$$

$[(A_n^{11})^{-1} A_j^{11}, \partial_*^{\alpha-e_n}] \partial_j u_I$  and  $[(A_n^{11})^{-1} A_j^{12}, \partial_*^{\alpha-e_n}] \partial_j u_{II}$ ,  $0 \leq j \leq n-1$ , belong to  $X_*^1([0, T]; \mathbf{R}_+^n)$ , and so does  $[(A_n^{11})^{-1} A_n^{12}, \partial_*^{\alpha-e_n}] \partial_n u_{II}$  because  $A_n^{12}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$ . Since  $(A_n^{11})^{-1} (F_I - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II}) \in X_*^{m-1}([0, T]; \mathbf{R}_+^n)$ , we have  $I^\alpha \in X_*^1([0, T]; \mathbf{R}_+^n)$ .

We express the next terms  $[A_j, \partial_*^\alpha] \partial_j u$ ,  $0 \leq j \leq n$ , as

$$(5.8_j) \quad - \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_j \partial_*^{\alpha-e_l} \partial_j u + G_j^\alpha.$$

Furthermore, by virtue of (5.6) the term  $\partial_*^{e_l} A_n \partial_*^{\alpha-e_l} \partial_n u$  can be rewritten as

$$(5.9) \quad - \sum_{j=0}^{n-1} \begin{pmatrix} \partial_*^{e_l} A_n^{11} (A_n^{11})^{-1} A_j^{11} & \partial_*^{e_l} A_n^{11} (A_n^{11})^{-1} A_j^{12} \\ 0 & 0 \end{pmatrix} u^{\alpha-e_l+e_j} \\ + x_n^{-1} \begin{pmatrix} 0 & \partial_*^{e_l} A_n^{12} - \partial_*^{e_l} A_n^{11} (A_n^{11})^{-1} A_n^{12} \\ \partial_*^{e_l} A_n^{21} & \partial_*^{e_l} A_n^{22} \end{pmatrix} u^{\alpha-e_l+e_n} + \begin{pmatrix} I_l^\alpha \\ 0 \end{pmatrix}$$

with

$$\begin{aligned} I_l^\alpha &= \sum_{j=0}^{n-1} \partial_*^{e_l} A_n^{11} [(A_n^{11})^{-1} A_j^{11}, \partial_*^{\alpha-e_l}] \partial_j u_I \\ &\quad + \sum_{j=0}^n \partial_*^{e_l} A_n^{11} [(A_n^{11})^{-1} A_j^{12}, \partial_*^{\alpha-e_l}] \partial_j u_{II} \\ &\quad + \partial_*^{e_l} A_n^{11} \partial_*^{\alpha-e_l} \left\{ (A_n^{11})^{-1} (F_I - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II}) \right\}. \end{aligned}$$

$I_l^\alpha$ ,  $0 \leq l \leq n$ , are shown to belong to  $X_*^1([0, T]; \mathbf{R}_+^n)$ , as  $I^\alpha$  is.  $G_j^\alpha$ ,  $0 \leq j \leq n-1$ , lie in  $X_*^1([0, T]; \mathbf{R}_+^n)$ . We have also  $G_n^\alpha \in X_*^1([0, T]; \mathbf{R}_+^n)$  because  $[A_n^{11}, \partial_*^\alpha] \partial_n u_I + \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_n^{11} \partial_*^{\alpha-e_l} \partial_n u_I$ ,  $[A_n^{21}, \partial_*^\alpha] \partial_n u_I + \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_n^{21} \partial_*^{\alpha-e_l} \partial_n u_I \in X_*^1(0, T; \mathbf{R}_+^n)$  by virtue of  $u_I \in X_{**}^{m-1}([0, T]; \mathbf{R}_+^n)$ , and  $[A_n^{12}, \partial_*^\alpha] \partial_n u_{II} + \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_n^{12} \partial_*^{\alpha-e_l} \partial_n u_{II}$ ,  $[A_n^{22}, \partial_*^\alpha] \partial_n u_{II} + \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_n^{22} \partial_*^{\alpha-e_l} \partial_n u_{II} \in X_*^1([0, T]; \mathbf{R}_+^n)$  by the fact that both  $A_n^{12}$  and  $A_n^{22}$  vanish on  $[0, T] \times \partial \mathbf{R}_+^n$ .  $[A_{n+1}, \partial_*^\alpha] u$  also belongs to  $X_*^1([0, T]; \mathbf{R}_+^n)$ .

All the matrices in (5.7), (5.8<sub>j</sub>),  $0 \leq j \leq n-1$ , (5.9) operating to the tangential derivatives  $u^\beta$ ,  $|\beta| = m-1$ , lie in  $\tilde{B}^\infty([0, T] \times \bar{\mathbf{R}}_+^n)$  because the matrices  $A_n^{12}$ ,  $A_n^{21}$  and  $A_n^{22}$  vanish on  $[0, T] \times \partial \mathbf{R}_+^n$ . Thus we can express  $J^\alpha$  like (5.5) with the function  $G^\alpha \in W_{1*}^1(0, T; \mathbf{R}_+^n)$  given by

$$G^\alpha = \alpha_n \begin{pmatrix} I^\alpha \\ 0 \end{pmatrix} - \sum_{l=0}^n \alpha_l \begin{pmatrix} I_l^\alpha \\ 0 \end{pmatrix} + \sum_{j=0}^n G_j^\alpha + [A_{n+1}, \partial_*^\alpha] u + \partial_*^\alpha F.$$

**The third step.** It is easy to see that the system (5.2) satisfies all the hypotheses in section 3. By (3.4),  $u^\alpha(0)$ ,  $|\alpha| = m-1$ , belong to  $\mathcal{H}_P^1(\mathbf{R}_+^n) \cap \widehat{\mathcal{H}}_Q^1(\mathbf{R}_+^n)$ . We apply Theorem 1 to obtain the solution  $(v^\alpha; |\alpha| = m-1)$  of (5.2) in the class  $\mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ . By the energy estimate we have

$$\begin{aligned} &e^{\lambda_0 t} |A_0(t)^{1/2} (u^\alpha(t) - v^\alpha(t))|_{L^2(\mathbf{R}_+^n)} \\ &\leq \int_0^t e^{\lambda_0 s} \left| A_0(s)^{-1/2} \left( J^\alpha(s) - \sum_{|\beta|=m-1} B^{\alpha\beta}(s) v^\beta(s) - G^\alpha(s) \right) \right|_{L^2(\mathbf{R}_+^n)} ds. \end{aligned}$$

Substituting (5.5) into this, we obtain

$$\begin{aligned} &e^{\lambda_0 t} |A_0(t)^{1/2} (u^\alpha(t) - v^\alpha(t))|_{L^2(\mathbf{R}_+^n)} \\ &\leq \sum_{|\beta|=m-1} |A_0^{-1/2} B^{\alpha\beta} A_0^{-1/2}|_{L^\infty} \int_0^t e^{\lambda_0 s} |A_0(s)^{1/2} (u^\beta(s) - v^\beta(s))|_{L^2(\mathbf{R}_+^n)} ds. \end{aligned}$$

Summing up the both sides for  $|\alpha| = m-1$ , we get by Gronwall's inequality

$$|A_0(t)^{1/2} (u^\alpha(t) - v^\alpha(t))|_{L^2(\mathbf{R}_+^n)} = 0, \quad |\alpha| = m-1,$$

that is,  $u^\alpha(t) = v^\alpha(t)$ ,  $0 \leq t \leq T$ . This proves  $u^\alpha \in \mathcal{X}_P^1([0, T]; \mathbf{R}_+^n)$ .

## 6. Regularity of normal derivatives.

In the previous section we proved the tangential regularity of solutions, that is,  $u \in X_{\tan}^m([0, T]; \mathbf{R}_+^n)$ . Since  $u_I \in X_{**}^{m-1}([0, T]; \mathbf{R}_+^n)$  by the assumption, we have  $u_I \in X_{\tan}^m([0, T]; \mathbf{R}_+^n) \cap X_{**}^{m-1}([0, T]; \mathbf{R}_+^n) = X_*^m([0, T]; \mathbf{R}_+^n)$ . From these facts we derive the regularity of the normal derivatives of  $u$ .

In this paper we only prove that

$$(6.1) \quad \partial_*^\alpha \partial_n^p u_I \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$$

for  $|\alpha| = \min\{m+1-2p, m-p\}$ ,  $0 \leq p \leq [(m+1)/2]$  and

$$(6.2) \quad \partial_*^\alpha \partial_n^p u_{II} \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$$

for  $|\alpha| = m-2p$ ,  $0 \leq p \leq [m/2]$ , which imply  $u_I \in Y_{**}^m(0, T; \mathbf{R}_+^n)$  and  $u_{II} \in Y_*^m(0, T; \mathbf{R}_+^n)$  respectively. The strong continuity in  $L^2$  of the derivatives will be shown in [23]. The following lemmata are crucial.

**Lemma 6.1.** *Suppose that  $1 \leq p \leq [(m+1)/2]$ . If*

$$\partial_*^\beta \partial_n^{p-1} u_{II} \in L^\infty(0, T; L^2(\mathbf{R}_+^n)), \quad |\beta| = m-2(p-1),$$

we have

$$\partial_*^\alpha \partial_n^p u_I \in L^\infty(0, T; L^2(\mathbf{R}_+^n)), \quad |\alpha| = m+1-2p.$$

**Lemma 6.2.** *Suppose that  $1 \leq p \leq [m/2]$ . If*

$$\partial_*^\beta \partial_n^p u_I \in L^\infty(0, T; L^2(\mathbf{R}_+^n)), \quad |\beta| = m+1-2p,$$

we have

$$\partial_*^\alpha \partial_n^p u_{II} \in L^\infty(0, T; L^2(\mathbf{R}_+^n)), \quad |\alpha| = m-2p.$$

We postpone the proofs of the lemmata and start the proof of (6.1) and (6.2). We proceed by induction with respect to the number  $p$ . When  $p = 0$ , (6.1) and (6.2) are nothing but the tangential regularity of  $u$ . Suppose that (6.1) and (6.2) are valid for  $p = q-1$  with  $1 \leq q \leq [m/2]$ . By the hypothesis of induction the assumption in Lemma 6.1 is satisfied with  $p = q$ . Hence (6.1) holds for  $p = q$ . This in turn implies the assumption in Lemma 6.2 with  $p = q$  and we have (6.2) for  $p = q$ . When  $m$  is even, the proof is completed. When  $m$  is odd, it follows from Lemma 6.1 that  $\partial_n^{[(m+1)/2]} u_I \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$  and this completes the proof.

**Proof of Lemma 6.1.** We operate  $\partial_*^\alpha \partial_n^{p-1}$  to (5.6) and express  $A_n^{11} \partial_*^\alpha \partial_n^p u_I$  as the

sum of the following terms:

$$(6.3) \quad -A_j^{11} \partial_*^\alpha \partial_n^{p-1} \partial_j u_I, \quad 0 \leq j \leq n-1,$$

$$(6.4) \quad -A_j^{12} \partial_*^\alpha \partial_n^{p-1} \partial_j u_{II}, \quad 0 \leq j \leq n,$$

$$(6.5) \quad [A_j^{11}, \partial_*^\alpha \partial_n^{p-1}] \partial_j u_I, \quad 0 \leq j \leq n,$$

$$(6.6) \quad [A_j^{12}, \partial_*^\alpha \partial_n^{p-1}] \partial_j u_{II}, \quad 0 \leq j \leq n-1,$$

$$(6.7) \quad [A_n^{12}, \partial_*^\alpha \partial_n^{p-1}] \partial_n u_{II},$$

$$(6.8) \quad \partial_*^\alpha \partial_n^{p-1} (F_I - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II}).$$

Since  $u_I \in X_*^m([0, T]; \mathbf{R}_+^n)$ , (6.3) and (6.5) belong to  $C^0([0, T]; L^2(\mathbf{R}_+^n))$ . The fact  $x_n^{-1} A_n^{12} \in \tilde{B}^\infty([0, T] \times \overline{\mathbf{R}_+^n})$  and the assumption imply (6.4)  $\in L^\infty(0, T; L^2(\mathbf{R}_+^n))$ . The term (6.6) lies in  $C^0([0, T]; L^2(\mathbf{R}_+^n))$ , and so does (6.7) because  $A_n^{12}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$ . It is easy to see that  $F_I - A_{n+1}^{11} u_I - A_{n+1}^{12} u_{II} \in X_*^{m-1}([0, T]; \mathbf{R}_+^n)$ . Thus we conclude  $\partial_*^\alpha \partial_n^p u_I \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$ . ■

**Proof of Lemma 6.2.** Abbreviating  $w^\alpha = \partial_*^\alpha \partial_n^p u_{II}$ ,  $|\alpha| = m - 2p$ , we prove  $w^\alpha \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$  by three steps. Noting that  $|\alpha| + p \leq m - 1$ , and hence the function  $w^\alpha$  is once differentiable, we first derive the equation

$$(6.9) \quad \sum_{j=0}^n A_j^{22} \partial_j w^\alpha + A_{n+1}^{22} w^\alpha = \sum_{|\beta|=m-2p} C^{\alpha\beta} w^\beta + H^\alpha \quad \text{in } [0, T] \times \mathbf{R}_+^n,$$

where  $C^{\alpha\beta}$  are elements of  $\tilde{B}^\infty([0, T] \times \overline{\mathbf{R}_+^n})$  with values in  $(l_0 - l_1) \times (l_0 - l_1)$  matrices, and  $H^\alpha$  is a  $\mathbf{C}^{l_0-l_1}$ -valued function in  $L^1(0, T; L^2(\mathbf{R}_+^n))$ . We remark that the matrix  $A_n^{22}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$ . Next, multiplying the equation (6.9) by such a weight  $\rho^{p+1}$  as the function  $\rho^{p+1} w^\alpha$  is sufficiently smooth up to the boundary, we derive the energy estimate for  $\rho^{p+1} w^\alpha$ . Finally, taking the limit along an appropriate sequence of  $\rho$ , we remove the weight from the estimate and then arrive at the conclusion  $w^\alpha \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$ .

**The first step.** It is easily verified that  $w^\alpha$  satisfies the equation

$$(6.10) \quad \sum_{j=0}^n A_j^{22} \partial_j w^\alpha + A_{n+1}^{22} w^\alpha = K^\alpha \quad \text{in } [0, T] \times \mathbf{R}_+^n$$

with

$$\begin{aligned} K^\alpha = & \alpha_n A_n^{22} \partial_*^{\alpha-e_n} \partial_n^{p+1} u_{II} + \sum_{j=0}^n [A_j^{22}, \partial_*^\alpha \partial_n^p] \partial_j u_{II} \\ & + [A_{n+1}^{22}, \partial_*^\alpha \partial_n^p] u_{II} - \partial_*^\alpha \partial_n^p \left( \sum_{j=0}^n A_j^{21} \partial_j u_I + A_{n+1}^{21} u_I \right) + \partial_*^\alpha \partial_n^p F_{II}. \end{aligned}$$

$K^\alpha$  is expressed as

$$(6.11) \quad \alpha_n x_n^{-1} A_n^{22} w^\alpha - \sum_{j=0}^{n-1} \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_j^{22} w^{\alpha-e_l+e_j} \\ - \sum_{l=0}^n \alpha_l x_n^{-1} \partial_*^{e_l} A_n^{22} w^{\alpha-e_l+e_n} - p \partial_n A_n^{22} w^\alpha + H^\alpha,$$

where  $H^\alpha$  is the sum of the following terms:

$$(6.12) \quad [A_j^{22}, \partial_*^\alpha \partial_n^p] \partial_j u_{II} \\ + \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_j^{22} \partial_*^{\alpha-e_l} \partial_n^p \partial_j u_{II}, \quad 0 \leq j \leq n-1,$$

$$(6.13) \quad [A_n^{22}, \partial_*^\alpha \partial_n^p] \partial_n u_{II} \\ + \sum_{l=0}^n \alpha_l \partial_*^{e_l} A_n^{22} \partial_*^{\alpha-e_l} \partial_n^p \partial_n u_{II} + p \partial_n A_n^{22} \partial_*^\alpha \partial_n^p u_{II},$$

$$(6.14) \quad - A_j^{21} \partial_*^\alpha \partial_n^p \partial_j u_I, \quad 0 \leq j \leq n,$$

$$(6.15) \quad [A_j^{21}, \partial_*^\alpha \partial_n^p] \partial_j u_I, \quad 0 \leq j \leq n-1,$$

$$(6.16) \quad [A_n^{21}, \partial_*^\alpha \partial_n^p] \partial_n u_I,$$

$$(6.17) \quad - A_{n+1}^{21} \partial_*^\alpha \partial_n^p u_I,$$

$$(6.18) \quad [A_{n+1}^{21}, \partial_*^\alpha \partial_n^p] u_I, \quad [A_{n+1}^{22}, \partial_*^\alpha \partial_n^p] u_{II},$$

$$(6.19) \quad \partial_*^\alpha \partial_n^p F_{II}.$$

All the matrices in (6.11) operating to  $w^\beta$ ,  $|\beta| = m - 2p$ , belong to  $\tilde{B}^\infty([0, T] \times \overline{\mathbf{R}_+^n})$  since the matrix  $A_n^{22}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$ . The terms (6.12) to (6.18) belong to  $L^\infty(0, T; L^2(\mathbf{R}_+^n))$ . As for (6.12) and (6.13) it follows from the fact that  $u_{II} \in X_*^{m-1}([0, T]; \mathbf{R}_+^n)$  and  $A_n^{22}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$ . Since  $x_n^{-1} A_n^{21} \in \tilde{B}^\infty([0, T] \times \overline{\mathbf{R}_+^n})$ , (6.14) belongs to  $L^\infty(0, T; L^2(\mathbf{R}_+^n))$  by the assumption. Since  $u_I \in X_*^m([0, T]; \mathbf{R}_+^n)$ , (6.15) belongs to  $C^0([0, T]; L^2(\mathbf{R}_+^n))$ , so does (6.16) because  $A_n^{21}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$ . Also (6.17) belongs to  $C^0([0, T]; L^2(\mathbf{R}_+^n))$ . Both the terms in (6.18) lie in  $C^0([0, T]; L^2(\mathbf{R}_+^n))$ . Thus  $w^\alpha$  satisfies the equation like (6.9).

**The second step.** Let  $\rho$  be a smooth function from  $[0, \infty)$  to  $[0, \infty)$  satisfying

$$(6.20) \quad 0 < \rho(r) \leq 1, \quad r > 0, \quad \rho(0) = 0, \quad 0 \leq r \rho'(r) \leq \rho(r).$$

Multiplying the both sides of (6.9) by the function  $\rho(x_n)^{p+1}$ , we have

$$\sum_{j=0}^n A_j^{22} \partial_j (\rho^{p+1} w^\alpha) + A_{n+1}^{22} (\rho^{p+1} w^\alpha) \\ = \frac{(p+1)\rho'}{\rho} A_n^{22} (\rho^{p+1} w^\alpha) + \sum_{|\beta|=m-2p} C^{\alpha\beta} (\rho^{p+1} w^\beta) + \rho^{p+1} H^\alpha.$$

The tangential regularity of  $u$  implies  $\rho^{p+1}w^\alpha \in X^1([0, T]; \mathbf{R}_+^n)$ . Hence we are led to the energy estimate

$$\begin{aligned}
& e^{\lambda_0 t} |\rho^{p+1} A_0^{22}(t)^{1/2} w^\alpha(t)|_{L^2(\mathbf{R}_+^n)} \\
& \leq |\rho^{p+1} A_0^{22}(0)^{1/2} w^\alpha(0)|_{L^2(\mathbf{R}_+^n)} \\
& \quad + (p+1) |x_n^{-1} A_0^{22-1/2} A_n^{22} A_0^{22-1/2}|_{L^\infty} \int_0^t e^{\lambda_0 s} |\rho^{p+1} A_0^{22}(s)^{1/2} w^\alpha(s)|_{L^2(\mathbf{R}_+^n)} ds \\
& \quad + \sum_{|\beta|=m-2p} |A_0^{22-1/2} C^{\alpha\beta} A_0^{22-1/2}|_{L^\infty} \int_0^t e^{\lambda_0 s} |\rho^{p+1} A_0^{22}(s)^{1/2} w^\beta(s)|_{L^2(\mathbf{R}_+^n)} ds \\
& \quad + \int_0^t e^{\lambda_0 s} |\rho^{p+1} A_0^{22}(s)^{-1/2} H^\alpha(s)|_{L^2(\mathbf{R}_+^n)} ds
\end{aligned}$$

with a constant  $\lambda_0$  satisfying

$$\frac{1}{2} A_0^{22}(t)^{-1/2} \left( A_{n+1}^{22}(t) + A_{n+1}^{22}(t)^* - \sum_{j=0}^n \partial_j A_j^{22}(t) \right) A_0^{22}(t)^{-1/2} \geq \lambda_0 I.$$

Here we use the fact that the matrix  $A_n^{22}$  vanishes on  $[0, T] \times \partial \mathbf{R}_+^n$  and so does the integration on the boundary. Summing up the above estimates for  $|\alpha| = m - 2p$  and putting

$$\begin{aligned}
\tilde{\mathbf{E}}_\rho(t) &= \sum_{|\alpha|=m-2p} |\rho^{p+1} A_0^{22}(t)^{1/2} w^\alpha(t)|_{L^2(\mathbf{R}_+^n)} \\
\tilde{\mathbf{F}}_\rho(t) &= \sum_{|\alpha|=m-2p} |\rho^{p+1} A_0^{22}(t)^{-1/2} H^\alpha(t)|_{L^2(\mathbf{R}_+^n)},
\end{aligned}$$

we have

$$e^{\lambda_0 t} \tilde{\mathbf{E}}_\rho(t) \leq \tilde{\mathbf{E}}_\rho(0) + N K_0^{-1} \int_0^t e^{\lambda_0 s} \tilde{\mathbf{E}}_\rho(s) ds + \int_0^t e^{\lambda_0 s} \tilde{\mathbf{F}}_\rho(s) ds$$

with a constant  $N$  independent of  $\rho$ . By Gronwall's inequality we get

$$(6.21) \quad \tilde{\mathbf{E}}_\rho(t) \leq \tilde{\mathbf{E}}_\rho(0) \exp(-\lambda_1 t) + \int_0^t \exp(-\lambda_1(t-s)) \tilde{\mathbf{F}}_\rho(s) ds$$

with  $\lambda_1 = \lambda_0 - N/K_0$ .

**The third step.** We choose a sequence of functions with the properties (6.20) monotone increasing and converging to 1 at each point  $r > 0$ . Since  $w^\alpha(0) \in L^2(\mathbf{R}_+^n)$  by (3.4), passing to the limit along the sequence of  $\rho$  in (6.21), we have  $w^\alpha(t) \in L^2(\mathbf{R}_+^n)$  and

$$\tilde{\mathbf{E}}(t) \leq \tilde{\mathbf{E}}(0) \exp(-\lambda_1 t) + \int_0^t \exp(-\lambda_1(t-s)) \tilde{\mathbf{F}}(s) ds, \quad 0 \leq t \leq T,$$

with

$$\begin{aligned}\tilde{E}(t) &= \sum_{|\alpha|=m-2p} |A_0^{22}(t)^{1/2} w^\alpha(t)|_{L^2(\mathbf{R}_+^n)} \\ \tilde{F}(t) &= \sum_{|\alpha|=m-2p} |A_0^{22}(t)^{-1/2} H^\alpha(t)|_{L^2(\mathbf{R}_+^n)}.\end{aligned}$$

This shows  $w^\alpha \in L^\infty(0, T; L^2(\mathbf{R}_+^n))$ . ■

## 7. Appendix.

**A. Mollifier.** Let  $\phi$  be a real valued  $C^\infty$ -function on  $\mathbf{R}^{n+1}$  with support contained in  $\{(x_0, x); 0 < x_0 < 1, |x| < 1, x_n > 0\}$  and

$$\int_{\mathbf{R}^{n+1}} \phi(y_0, y) dy_0 dy = 1, \quad \phi \geq 0.$$

Let  $a, b$  and  $\varepsilon_0$  be constants with  $0 < \varepsilon_0 < b - a$ . Let  $1 \leq p \leq \infty$ . We define the linear operator  $\mathcal{M}_\varepsilon$ ,  $0 < \varepsilon < \varepsilon_0$ , from  $L^p(a, b; L^2(\mathbf{R}_+^n))$  to  $L^p(a, b - \varepsilon_0; L^2(\mathbf{R}_+^n))$  by

$$\mathcal{M}_\varepsilon u(x_0, x', x_n) = \int_0^1 \int_{\mathbf{R}_+^n} \phi(y_0, y', y_n) u(x_0 + \varepsilon y_0, x' + \varepsilon y', x_n e^{\varepsilon y_n}) dy_0 dy' dy_n.$$

The operator  $\mathcal{M}_\varepsilon$  was introduced by Rauch [16] in the study of first order systems with boundary characteristics. The operation of the mollifier has smoothing effects in the following sense.

### Lemma A.0.

(1) Let  $u \in W_p^m(a, b; \mathbf{R}_+^n)$  (resp.  $W_{p*}^m(a, b; \mathbf{R}_+^n)$ ,  $W_{p**}^m(a, b; \mathbf{R}_+^n)$ ),  $1 \leq p < \infty$ ,  $m \in \mathbf{Z}_+$ . Then,  $\partial_{\tan}^\alpha \mathcal{M}_\varepsilon u \in X^m([a, b - \varepsilon_0]; \mathbf{R}_+^n)$  (resp.  $X_*^m([a, b - \varepsilon_0]; \mathbf{R}_+^n)$ ,  $X_{**}^m([a, b - \varepsilon_0]; \mathbf{R}_+^n)$ ) for any  $\alpha \in \mathbf{Z}_+^{n+1}$ . We have

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \mathcal{M}_\varepsilon u &= u \quad \text{in } W_p^m(a, b - \varepsilon_0; \mathbf{R}_+^n) \\ &\quad (\text{resp. } W_{p*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n), W_{p**}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)).\end{aligned}$$

The assertions are valid when we replace  $W_p^m(I; \mathbf{R}_+^n)$ ,  $W_{p*}^m(I; \mathbf{R}_+^n)$  and  $W_{p**}^m(I; \mathbf{R}_+^n)$  with  $X^m(\bar{I}; \mathbf{R}_+^n)$ ,  $X_*^m(\bar{I}; \mathbf{R}_+^n)$  and  $X_{**}^m(\bar{I}; \mathbf{R}_+^n)$  respectively.

(2) Let  $u \in W_p^m(a, b; \mathbf{R}_+^n)$  (resp.  $W_{p**}^m(a, b; \mathbf{R}_+^n)$ ),  $1 \leq p \leq \infty$ ,  $m \in \mathbf{N}$ . We assume that  $\gamma_0[u] = 0$  holds in  $L^p(a, b; H^{m-1/2}(\partial \mathbf{R}_+^n))$ . Then, we have  $\gamma_0[\partial_{\tan}^\alpha \mathcal{M}_\varepsilon u] = 0$  in  $C^\infty([a, b - \varepsilon_0] \times \partial \mathbf{R}_+^n)$  for any  $\alpha \in \mathbf{Z}_+^{n+1}$ .

We list several properties of commutators between first order differential operators and the mollifier. For the proofs see [23]. In what follows we assume  $1 \leq p < \infty$ .

**Lemma A.1.** Let  $A \in \tilde{B}^\infty([a, b] \times \overline{\mathbf{R}_+^n})$ .

(1) Let  $\partial = \partial_0, \dots, \partial_{n-1}$  and  $u \in W_{p^*}^m(a, b; \mathbf{R}_+^n)$ ,  $m \in \mathbf{N}$ . Then,  $[A\partial, \mathcal{M}_\varepsilon]u \in W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)$ ,  $0 < \varepsilon < \varepsilon_0$ . There exists a constant  $C$  independent of  $A$ ,  $u$  and  $\varepsilon$  such that

$$|[A\partial, \mathcal{M}_\varepsilon]u|_{W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)} \leq C|A|_{\tilde{B}^m([a, b] \times \overline{\mathbf{R}_+^n})}|u|_{W_{p^*}^m(a, b; \mathbf{R}_+^n)}.$$

Moreover, we have

$$\lim_{\varepsilon \rightarrow 0} [A\partial, \mathcal{M}_\varepsilon]u = 0 \quad \text{in } W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n).$$

(2) Let  $u \in W_{p^{**}}^m(a, b; \mathbf{R}_+^n)$ ,  $m \in \mathbf{N}$ . Then,  $[A\partial_n, \mathcal{M}_\varepsilon]u \in W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)$ ,  $0 < \varepsilon < \varepsilon_0$ . There exists a constant  $C$  independent of  $A$ ,  $u$  and  $\varepsilon$  such that

$$|[A\partial_n, \mathcal{M}_\varepsilon]u|_{W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)} \leq C|A|_{\tilde{B}^m([a, b] \times \overline{\mathbf{R}_+^n})}|u|_{W_{p^{**}}^m(a, b; \mathbf{R}_+^n)}.$$

Moreover, we have

$$(7.1) \quad \lim_{\varepsilon \rightarrow 0} [A\partial_n, \mathcal{M}_\varepsilon]u = 0 \quad \text{in } W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n).$$

**Lemma A.2.** Let  $A \in \tilde{B}^\infty([a, b] \times \overline{\mathbf{R}_+^n})$  and  $u \in W_{p^*}^m(a, b; \mathbf{R}_+^n)$ ,  $m \in \mathbf{N}$ . We assume that  $A|_{[a, b] \times \partial \mathbf{R}_+^n} = 0$ . Then,  $[A\partial_n, \mathcal{M}_\varepsilon]u \in W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)$ ,  $0 < \varepsilon < \varepsilon_0$ . There exists a constant  $C$  independent of  $A$ ,  $u$  and  $\varepsilon$  such that

$$|[A\partial_n, \mathcal{M}_\varepsilon]u|_{W_{p^*}^m(a, b - \varepsilon_0; \mathbf{R}_+^n)} \leq C|A|_{\tilde{B}^{m \vee 2}([a, b] \times \overline{\mathbf{R}_+^n})}|u|_{W_{p^*}^m(a, b; \mathbf{R}_+^n)}.$$

The assertion in (7.1) is valid also in this case.

**B. Weak convergence of functions.** Let  $X_j$ ,  $0 \leq j \leq m$ , be Hilbert spaces with  $X_j$  continuously embedded to  $X_{j-1}$ ,  $1 \leq j \leq m$ . We assume that  $X_j$ ,  $1 \leq j \leq m$ , are dense in  $X_0$ .

**Lemma B.** Let  $I$  be a finite open interval and  $m \in \mathbf{N}$ .

(1) If  $u \in \bigcap_{j=0}^m W_\infty^{m-j}(I; X_j)$ , then  $\partial^{m-j}u \in C_w^0(\bar{I}; X_j)$ ,  $1 \leq j \leq m$ .

(2) Let  $\{u_k\}$  be a bounded sequence in  $\bigcap_{j=0}^m W_\infty^{m-j}(I; X_j)$ . There exists a subsequence  $\{u_{k_p}\}$  and  $u \in \bigcap_{j=0}^m W_\infty^{m-j}(I; X_j)$  such that

$$\lim_{p \rightarrow \infty} \partial^{m-j}u_{k_p}(t) = \partial^{m-j}u(t) \text{ weakly in } X_j \text{ uniformly on } \bar{I}, \quad 1 \leq j \leq m.$$

*Proof:* By using the mollifier an element of  $\bigcap_{j=0}^m W_\infty^{m-j}(I; X_j)$  is approximated by a sequence in  $C^\infty(\bar{I}; X_m)$  which is bounded in  $\bigcap_{j=0}^m W_\infty^{m-j}(I; X_j)$  and converges to the element in  $\bigcap_{j=0}^m W_1^{m-j}(I; X_j)$ . Therefore it suffices to show (2) under the additional condition

$$\partial^{m-j}u_k \in C_w^0(\bar{I}; X_j), \quad 1 \leq j \leq m.$$



Since the dual space of  $X_0$  is dense in that of  $X_j$  ([24], Chapter 2), it is proved that the sequences  $\{\partial^{m-j}u_k\}$ ,  $1 \leq j \leq m$ , are equicontinuous in the weak topology of  $X_j$ . Thanks to the local weak compactness and the weak completeness of Hilbert spaces we can choose, by Ascoli–Arzéla argument, a subsequence  $\{u_{k_p}\}$  so that  $\{\partial^{m-j}u_{k_p}(t)\}$ ,  $1 \leq j \leq m$ , converge weakly in  $X_j$  uniformly on  $\bar{I}$ . The limits  $v_j(t)$  define functions in  $C_w^0(\bar{I}; X_j)$ .  $v_j$  are uniformly Lipschitz continuous functions on  $\bar{I}$  with values in  $X_{j-1}$  and hence lie in  $W_\infty^1(I; X_{j-1})$ . We put  $u = v_m \in C_w^0(\bar{I}; X_m)$ . It is verified that  $\partial^{m-j}u = v_j$ ,  $1 \leq j \leq m$ . Since  $\partial^{m-1}u \in W_\infty^1(I; X_0)$ , we have  $u \in \bigcap_{j=0}^m W_\infty^{m-j}(I; X_j)$ . This completes the proof. ■

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